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STAGING SOLID-PROPELLANT ROCKETS FOR SPACECRAFT RETARDATION

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I. INTRODUCTION

Within the next decade several attempts will be made to orbit or land on the Moon, Venus or Mars. For each of these missions the incoming spacecraft will have to be retarded to some desired velocity. The propulsion system required to do this job will have to be rugged, efficient, compact, and reliable over long periods of time and temperature changes. The present answer to the problems posed by these requirements lies in solid-rocket propulsion.

Many missions will have low-energy trajectories. These trajectories, in turn, will require low-velocity decrements for orbiting and landing, with a resulting low staging efficiency. Nearly all the lunar trajectories fall in this low-energy class. On the other hand, there will be some planetary missions in which low-transit-time, high-energy trajectories will be advantageous because of restrictions on times-of-flight. Velocity decrements could be as high as 25,000 ft/sec on such high-energy missions. It is in this class of missions that staging efficiency is high.

This study presents three staging schemes, the weight of resulting payload being the dependent variable. The prime disadvantage of optimum staging is the requirement of different-sized motors for each stage which results in a development program for each motor. The purpose of this study is to find out how far from the optimum point certain other staging schemes would be,when these schemes are less expensive, less development time consuming, and perhaps more reliable.

Two of these alternate schemes of staging spacecraft seem competitive with optimum staging, when the loss in payload is balanced by the gain in development time and money. In the first scheme motors of identical size are used in each stage; in the second scheme an off-the-shelf high-performance motor of the required weight range in the staging sequence is used, thereby requiring one less motor to be developed in the propulsion system.

[1]This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract NASw-6, sponsored by the National Aeronautics and Space Administration.

II. ASSUMPTIONS

The following assumptions are made:

1. The equations of motion described here are assumed to represent a drag-free condition. Gravity–burning time losses can be approximated by nominal values of g and t_b to be experienced.

2. The stages in each described sequence have equal values of specific impulse and structural factor. The equations are presented in general form; however, to avoid complicated equations and results and to minimize the number of assumptions for rocket motor characteristics, all stages are assumed to be of equal performance.

III. STAGING ANALYSES

A. Optimum Staging

For any I_s per stage, the resulting velocity attained in an n staged configuration is

$$\Sigma V = c_1 \ln \left(\frac{1}{1 - \zeta_1}\right) + c_2 \ln \left(\frac{1}{1 - \zeta_2}\right) + \cdots + c_n \ln \left(\frac{1}{1 - \zeta_n}\right) \tag{1}$$

where

$$\Sigma V = (\Delta V_1 + g t_{b1}) + (\Delta V_2 + g t_{b2}) + \cdots + (\Delta V_n + g t_{bn}), \text{ and } \zeta_i = \frac{W_{pi}}{W_{0i}}$$

Raising both sides of (1) to e yields

$$\exp \Sigma V = \prod_{k=1}^{r} \left(\frac{1}{1 - \zeta_k}\right)^{c_k} \tag{2}$$

which is the general equation for a staged rocket system.

By subtracting all of the stages from the initial gross weight, another important formula is obtained:

$$\frac{W_{pl}}{W_{01}} = 1 - A_1 X_1 - A_2 X_2 - A_3 X_3 - \cdots - A_n X_n \tag{3}$$

where

$$A_k = \frac{W_{pi} + W_{mi}}{W_{pi}}; \qquad X_i = \frac{W_{pi}}{W_{01}}$$

Expanding X_i yields

$$\left.\begin{aligned}
X_1 &= \frac{W_{p1}}{W_{01}} = \zeta_1 \\
X_2 &= \frac{W_{p2}}{W_{01}} = \zeta_2 \ [1 - A_1X_1] \\
X_3 &= \frac{W_{p3}}{W_{01}} = \zeta_3 \ [1 - A_1X_1 - A_2X_2] \\
X_n &= \frac{W_{pn}}{W_{01}} = \zeta_n \ [1 - A_1X_1 - A_2X_2 - \cdots - A_{n-1}\,X_{n-1}]
\end{aligned}\right\} \tag{4}$$

When the equations of (4) are substituted into (3), and the terms are rearranged, there results:

$$\frac{W_{pl}}{W_{01}} = \prod_{k=1}^{n} (1 - A_k\ \zeta_k) \tag{5}$$

Equation (5), when used in conjunction with (2), will give resulting payloads for any known mission and known rocket parameters. Assumptions of $\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_{n-1}$ must be made to determine ζ_n (from 2) and W_{pl}/W_{01} (from 5). When I_s and A are assumed equal for all stages, equations (2) and (5) become

$$\exp \frac{\Sigma V}{c} = \prod_{k=1}^{n} \left(\frac{1}{1 - \zeta_k} \right) \tag{2a}$$

$$\frac{W_{pl}}{W_{01}} = \prod_{k=1}^{n} (1 - A\ \zeta_k) \tag{5a}$$

The optimum value of W_{pl}/W_{01} for equal performance stages is found when

$$\frac{\partial \left(\frac{W_{pl}}{W_{01}} \right)}{\partial \zeta_1} = 0, \quad \frac{\partial \left(\frac{W_{pl}}{W_{01}} \right)}{\partial \zeta_2} = 0, \cdots, \quad \frac{\partial \left(\frac{W_{pl}}{W_{01}} \right)}{\partial \zeta_n} = 0 \tag{6}$$

and is

$$\zeta_1 = \zeta_2 = \zeta_3 = \cdots = \zeta_n = 1 - \exp - \frac{\Sigma V}{nc} \tag{7}$$

Equation (5a) then reduces to

$$\left(\frac{W_{pl}}{W_{01}} \right)_{opt} = (1 - A\ \zeta_{opt})^n \tag{8}$$

One more step can be taken in this analysis. If $n \to \infty$ in (8), the $(W_{pl}/W_{01})_{opt}$ converges to a limit which is the maximum value of the payload for an infinite number of stages. As expected, this maximum is a function of ΣV alone, for given values of I_s and A:

$$\left.\frac{W_{pl}}{W_{01}}\right|_{max} = \exp - \frac{A}{c} \Sigma V \tag{9}$$

Equation (9) can also be derived from Newton's second law. The derivation is found in an appendix.

In Fig. 1 (which is a plot of a two-stage vehicle configuration) the solutions of Eq. (2a) and 5a) are shown for three velocity increments, illustrating the flatness or sharpness of the optimum as the velocity increment is decreased or increased. Figure 2, which is a plot of a three-stage vehicle configuration, shows the variation in the payload ratio for various assumed values of ζ_1 and ζ_2. The dashed line through the maximum points of the curves is the solution of the equation

$$\left[\frac{\partial\left(\dfrac{m_{pl}}{m_{01}}\right)}{\partial \zeta_1}\right]_{\zeta_2 = \text{const}} = 0$$

and is derived from (2a) and (5a) when $n = 3$. The maximum payload for an infinite n is seen to be the upper limit of a straight line with a slope $-(A/c)$, as defined by (9) and illustrated in Fig. 4.

B. Staging with Motors of Equal Sizes

For the case of motors of equal size (which implies equal weights of propellant for each stage), Eq. (2) remains the same. The counterpart of Eq. (5) is derived from:

$$W_{01} = W_{pl} + W_p A_1 + W_p A_2 + W_p A_3 + \cdots + W_p A_n$$

and by dividing both sides by W_{01} and transposing, there results

$$\frac{W_{pl}}{W_{01}} = 1 - \zeta_1 \sum_{k=1}^{n} A_k \tag{10}$$

Equation (2a) also remains unchanged for all methods of staging (as long as propellants of equal specific impulse are used in all stages). However, the values of ζ_k that are substituted in (2a) may vary. In the case of motors of equal size

$$\left.\begin{aligned} \frac{1}{\zeta_1} &= \frac{1}{\zeta_2} + A_1 \\ \frac{1}{\zeta_2} &= \frac{1}{\zeta_3} + A_2 \\ \frac{1}{\zeta_i} &= \frac{1}{\zeta_{i+1}} + A_i \end{aligned}\right\} \tag{11}$$

so that, in general, any ζ_k, by successive substitutions of (11), can be expressed in terms of ζ_1 and A_k, and is

$$\frac{1}{\zeta_k} = \frac{1}{\zeta_1} - \sum_{i=1}^{k-1} A_i \tag{12}$$

As in the case for optimum staging, equal structural factors (A_k) for each stage are also assumed here, in which case (12) reduces to

$$\frac{1}{\zeta_k} = \frac{1}{\zeta_1} - (k - 1)\,A \tag{12a}$$

When the values for ζ_2, ζ_3, ζ_4, ····, ζ_n are found by either (12) or (12a), and these values are substituted into (2a), a polynomial in ζ_1 is formed, the degree of which is n. When this equation is solved for the real positive root of ζ_1, where $0 < \zeta_1 < 1$ (first approximation $\zeta_1 \approx 1/An$), the payload ratio is determined through (10). The results of staging of motors of equal size are shown on the same curves as the results for optimum staging.

C. Staging with Motors of Odd Sizes

Even though staging with odd-size motors could be more efficient than staging with motors of equal size, a rigorous mathematical analysis without certain assumptions is both difficult and unnecessary. A qualitative discussion can suffice,using the results of Sections III-A and III-B as examples. Figure 1 illustrates the change in payload ratio with ζ_1 for a two-stage vehicle. Over a large portion of the curves for all three velocities shown, the payload ratio varies little with ζ_1. In fact, for a retarded velocity of 15,000 ft/sec, the payload ratio decreases less than 2½% between the ranges $\zeta_1 = 0.3$ to $\zeta_1 = 0.7$. The relationship of motors of equal size, which is a special case of odd-size motors, is shown with respect to the optimum point of the curves. It is seen that it is possible for an odd-size motor to lie closer to the optimum than the motor of equal size. This suggests that if an off-the-shelf motor were available which satisfied the required weight and performance criteria, it could be used in the staging sequence with little loss in payload. In such a case, one less stage would have to be developed for the propulsion system, and this stage, through previous qualification testing and use, could prove to be extremely reliable.

IV. FACTORS AFFECTING STAGING

A. Improved Motor Performance

As propellant specific impulses increase and the inert weights of the motor and the associated structure decrease with future improvements, staging efficiency decreases. Higher performance propellants, new nozzle and case materials, as well as improved grain designs resulting in higher propellant mass fractions will make high-performance single staging comparable to multi-staging, when reliability is taken into consideration.

Figure 3 illustrates the decrease of staging efficiency as motor performance improves. The abscissa in Fig. 3 can be taken to be a time coordinate, indicating that in a few more years staging might not give the payload advantages we know today.

B. Reliability of Staged Rocket Configurations

It is difficult to foresee the exact reliability demands that will be made on the propulsion system for a spacecraft. However, it is apparent that they will be high. With present solid-propellant quality control, it is possible to demonstrate a motor reliability of 90% with a 95% confidence level through a qualification test program of fewer than 30 motors. This demonstration, together with past experience in which many well designed units have shown reliabilities of greater than 99%, should inspire confidence in mission capability. Although this cannot be used as a criterion for qualification, this knowledge in conjunction with favorable nondestructive test data will be taken into consideration. A more serious reliability difficulty is found within the staging mechanisms and separation devices themselves. Qualitatively, it can be stated that no serious problem exists here that has not been overcome in the past. Quantitatively one can cite the successful qualification programs of the *Juno I*, *Juno II*, and *Nike-Zeus* missile systems, all of which contain a three-stage solid-propellant sequence. These vehicles attained a high separation reliability with a minimum of qualification.

V. CONCLUSIONS

In an analysis of staging, one reaches the point of diminishing returns when the increase in payload is balanced against the increase in the number of stages. Considering the increase in complexity and the decrease in reliability vs the small increase in payload weight as the number of stages in a vehicle configuration increases, it is felt that no more than three solid-propellant stages would ever be employed for retarding a spacecraft for either lunar or planetary use. In Fig. 4, it is shown that staging does not become overly advantageous until a velocity of approximately 10,000 ft/sec is reached. When the required velocity is approximately 15,000 ft/sec, however, staging becomes apparent and motors of either odd or equal size could be employed with little loss in payload.

If the velocity requirement is greater than 15,000 ft/sec, it is possible that optimum staging would be desired. If such a case exists, it would be of great benefit to use solid-propellant scaling techniques which would eliminate the need for designing separate motors for each stage. However, a separate qualification test program for each motor would still have to be made.

NOMENCLATURE

A	structural factor $= 1 + \dfrac{W_m}{W_p}$
c	exhaust velocity
g	gravitational constant
I_s	propellant specific impulse
m	mass
t_b	burning time
V	velocity
W	weight
ζ	propellant mass fraction

Subscripts

m	inert components of motor
p	propellant
pl	payload
$1, 2, \cdots, i, k, n$	stage numbers
0	initial conditions

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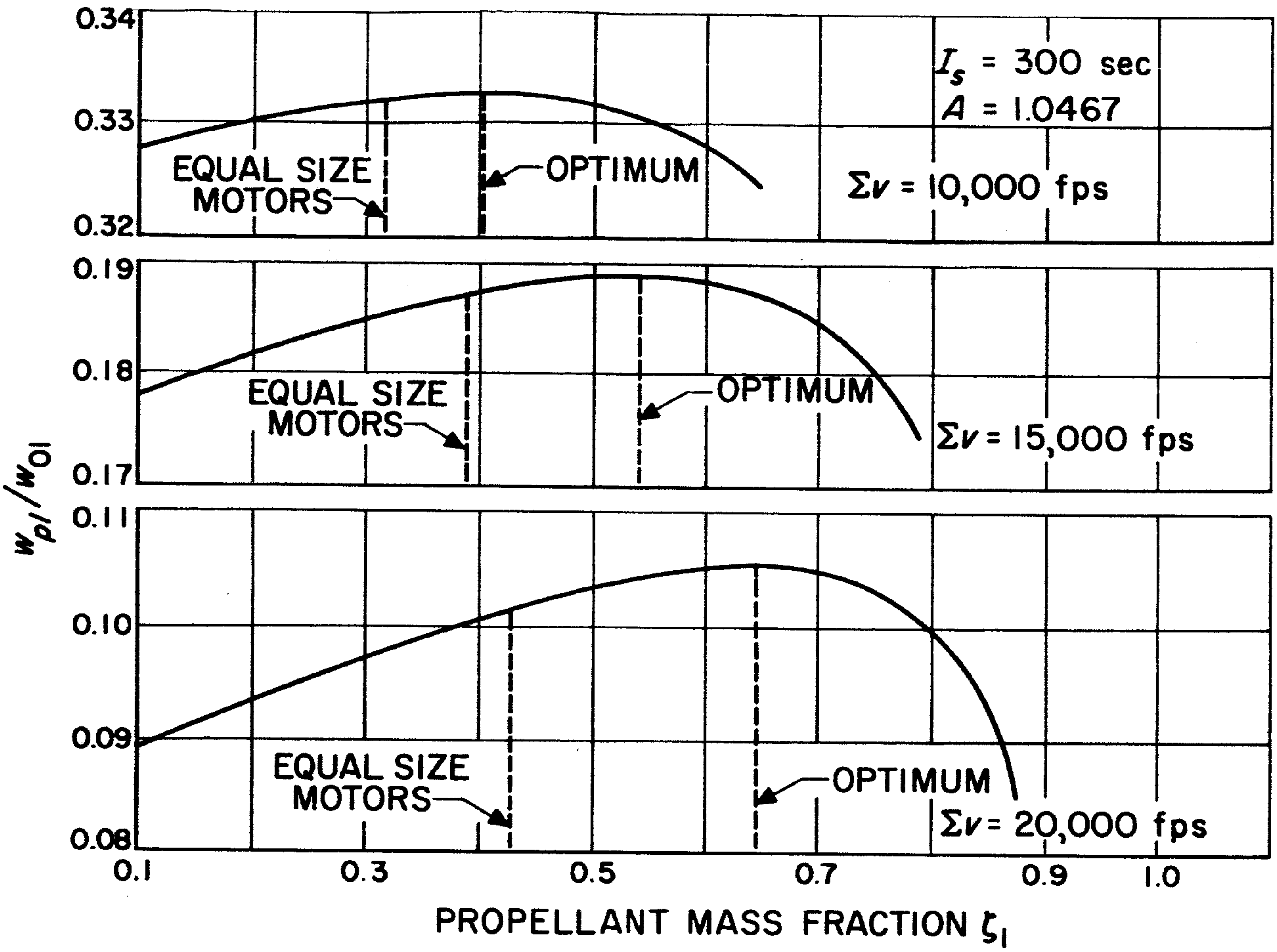

Fig. 1. Plot of ζ_1 vs W_{pl}/W_{01} for three velocity increments for a two-stage configuration

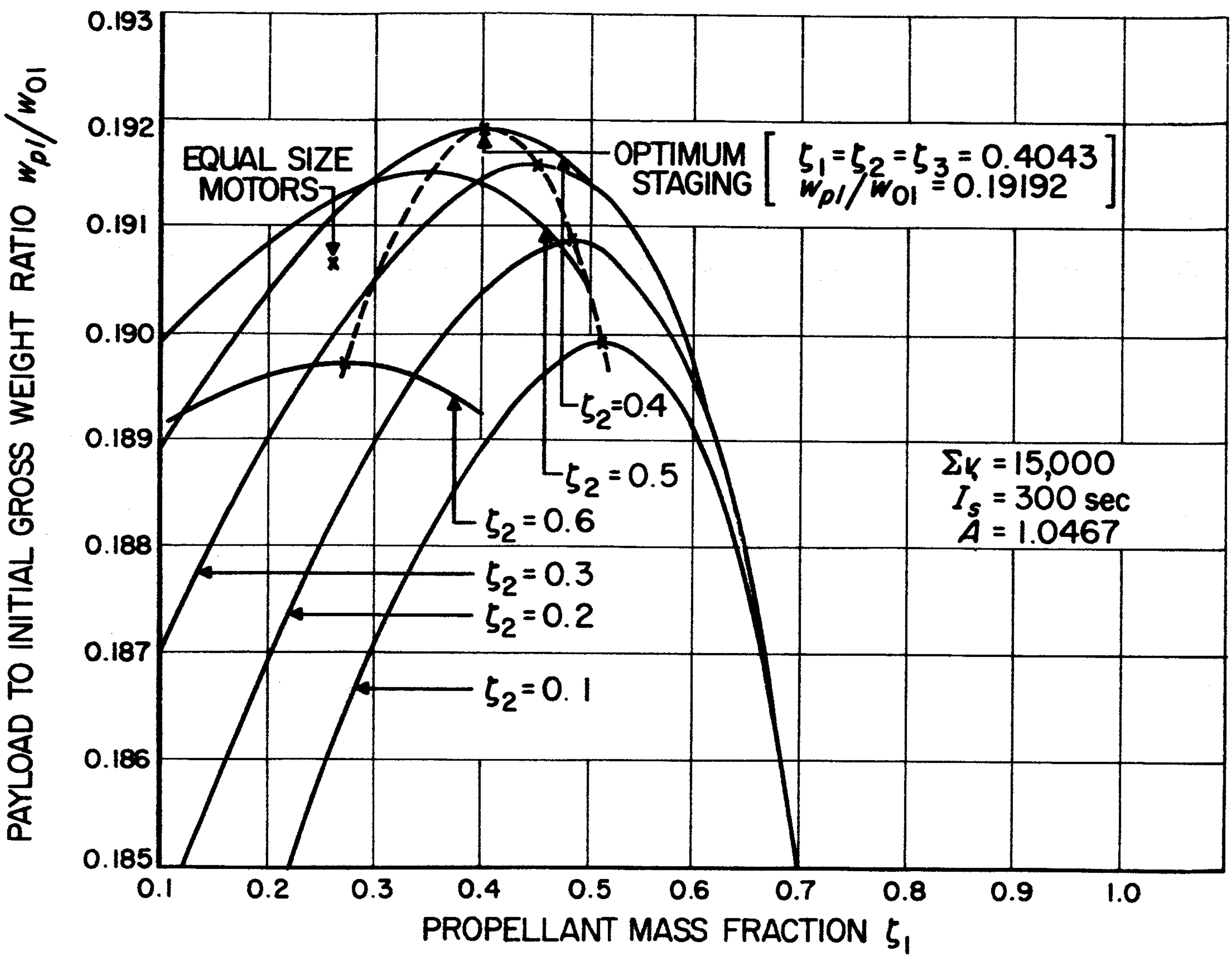

Fig. 2. Plot of ζ_1 vs W_{pl}/W_{01} for several ζ_2 for a three-stage configuration

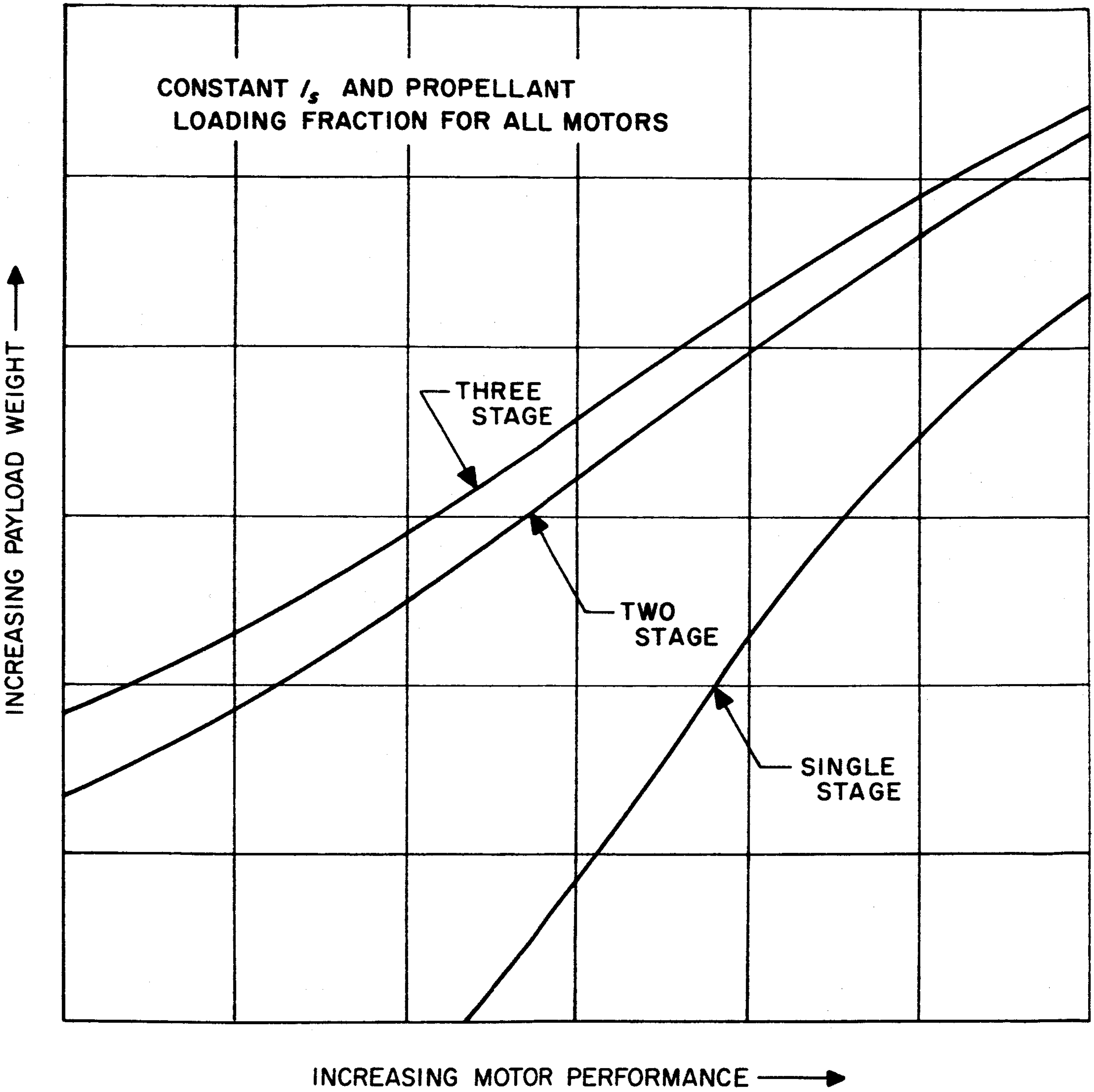

Fig. 3. Staging efficiency vs improving motor performance

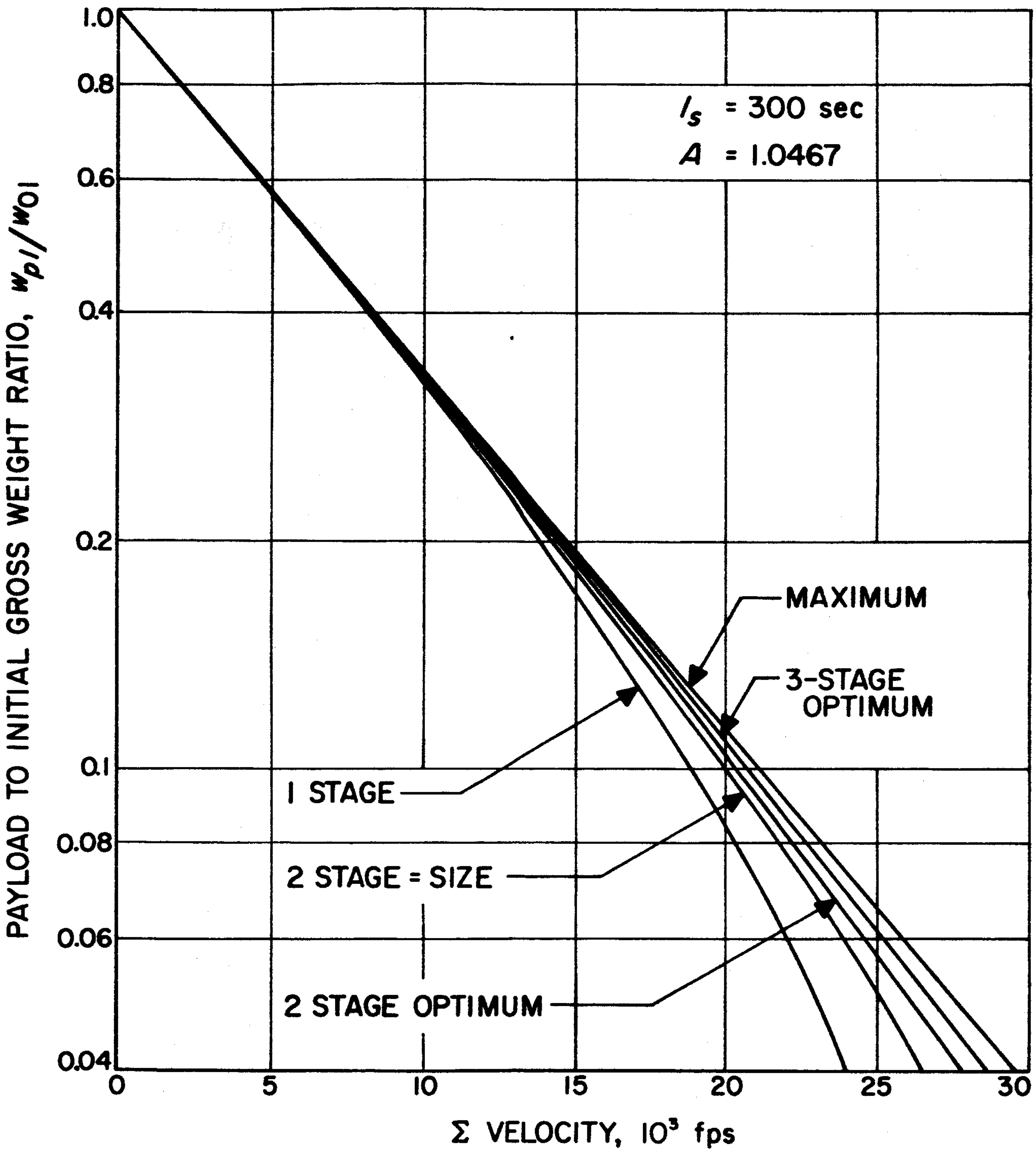

Fig. 4. Velocity vs W_{pl}/W_{01} for several staging configurations

APPENDIX

Mass Derivation for an Infinitely Staged Rocket

Equating forces that are acting on the vehicle at any time,

$$m_{vi}\ a_i = c\ \dot{m}_p \tag{A-1}$$

where

m_{vi} is the instantaneous mass of vehicle

a_i is the instantaneous acceleration of vehicle

$\dot{m}_p$ is the instantaneous propellant mass flow rate

$$\frac{dV_i}{dt} = \frac{c}{m_{vi}} \frac{dm_p}{dt}$$

$$dV_i = c\ \frac{dm_p}{m_{vi}} \tag{A-2}$$

An infinitely staged rocket is defined by letting an incremental piece of inert material be dropped off with each incremental piece of propellant; i.e.,

$$m_{vi} = m_0 - Ax \tag{A-3}$$

where $x = f(m_p, t)$ and is defined as the amount of expended propellant. In taking the limit of an "n" staged vehicle as $n \to \infty$, (3) can also be visualized. As the number of stages increases, the size of the motor and the inert weight for each stage decreases, until the limit is reached at which dx amount of propellant is burned and $(A - 1)\, dx$ amount of inert material is discarded $[dx + (A - 1)\, dx = A\, dx]$. In (2), dm_p can be represented as the rate of propellant increase or decrease; hence

$$dV_i = c \frac{dx}{m_0 - Ax} \tag{A-4}$$

Upon integrating, there results

$$V = -\frac{c}{A} \ln (m_0 - Ax) \Big/_{x=0}^{x=m_p}$$

The upper limit $(m_0 - Am_p)$ is just the resulting payload (m_{pl}); thus

$$V = -\frac{c}{A} \ln \frac{m_{pl}}{m_0} \tag{A-5}$$

Raising both sides of (5) to e:

$$\exp -\frac{A}{c} V = \frac{m_{pl}}{m_0} \tag{A-6}$$

Equation (A-6) is the maximum payload that can result by staging an infinite number of stages.

Two interesting conclusions can be reached by this derivation and its results:

1. If one knows the specific impulse and structural factor of a system of rockets that are to be optimally staged, the approximate weight of the payload can be easily determined. For low

values of retarded velocity (< 10,000 ft/sec), this maximum is a very close approximation to even single staging.

2. A method for incorporating the mass of ablative materials burned out during the course of motor burning is found in (2). If $A' = 1 + \mu'$ is some time function of inert material expenditure rate, the resulting value for the payload mass can be calculated easily by integration.

In either (A-1) or (A-2), the gt_b losses can be easily taken into account by insertion of the gravity term in Eq. (A-1). However, if g is variable, a numerical integration is necessary.